# A suggested lacunarity expression for Sierpinski carpets



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LETTER TO THE EDITOR

# A suggested lacunarity expression for Sierpinski carpets†

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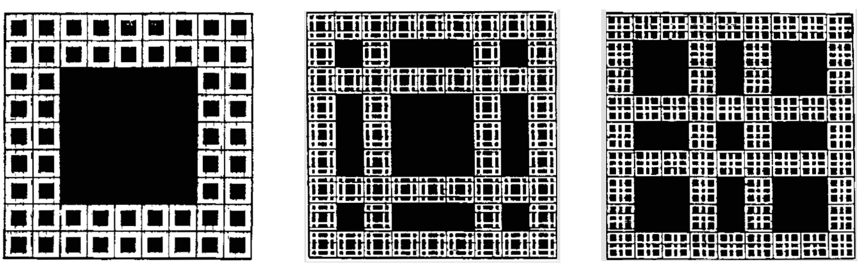
**Abstract.** A suggested approximate expression for lacunarity is proposed. Some numerical investigations show that the old expression fails in describing the homogeneity of elimination for some Sierpinski carpets.

The lacunarity $L$ is a geometric parameter of a fractal. It measures the extent of the failure of a fractal to be translationally invariant [1–3]. There can be several fractals with the same fractal dimensionality $D$ but different lacunarities, reflecting that the eliminated squares are scattered differently (see figure 1 and table 1). More recent study shows that lacunarity plays an important role in the study of critical phenomena on fractals. It may be a parameter classifying the universal class [5]. In this letter we discuss the lacunarity expression for Sierpinski carpets.

Following Mandelbrot and Gefen *et al* [5, 6], we construct Sierpinski carpets in the following way: consider a square of unit area and subdivide it into $b^2$ subsquares, out of which $l^2$ subsquares are cut out. These eliminated subsquares may be condensed or scattered corresponding to the less homogeneous (high lacunarity) or more homogeneous (low lacunarity) case. At the limit of zero lacunarity, a fractal becomes a translationally invariant lattice from a self-similar structure and then the physical properties of the fractal become identical to those of an abstract analytically continued hypercubic lattice. To measure the deviation of a fractal from being translationally invariant, the lacunarity parameter $L$ is used. An approximate expression is given by



(*a*) (*b*) (*c*)

**Figure 1.** Three two-stage Sierpinski carpets with the same fractal dimensionality $D = \ln(9^2 - 5^2)/\ln 9 = 1.832$ (the same $b$, $l$ values: $b = 9$, $l = 5$) but different lacunarity. (*a*) Central cutout carpet, the least homogeneity, $L = 0.1176$ ($L' = 0.385$), (*b*) $L = 0.0544$ ($L' = 0.176$), (*c*) $L = 0.0402$ ($L' = 0$).

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**Table 1.** Values of $\bar{n}$, $L'(s)$, $L(s)$ and $L$ for three Sierpinski carpets (figure 1) with the same $b$, $l$ values ($b=9$, $l=5$) but different ways of elimination.

| | Figure 1(*a*) | | | Figure 1(*b*) | | | Figure 1(*c*) | | |
|---|---|---|---|---|---|---|---|---|---|
| $S$ | $\bar{n}$ | $L'(s)$ | $L(s)$ | $\bar{n}$ | $L'(s)$ | $L(s)$ | $\bar{n}$ | $L'(s)$ | $L(s)$ |
| 2 | 2.473 | 0.621 | 0.664 | 2.437 | 0.621 | 0.323 | 2.437 | 0.621 | 0.323 |
| 3 | 4.408 | 6.854 | 0.594 | 5.551 | 2.982 | 0.311 | 5.551 | 0.860 | 0.167 |
| 4 | 7.000 | 12.444 | 0.504 | 8.889 | 3.210 | 0.202 | 10.556 | 2.469 | 0.148 |
| 5 | 10.560 | 16.486 | 0.385 | 13.440 | 5.606 | 0.176 | 16 | 0† | 0 |
| 6 | 15.750 | 10.188 | 0.203 | 20 | 0 | 0 | 23.750 | 6.187 | 0.105 |
| 7 | 24 | 0 | 0 | 30.222 | 8.395 | 0.096 | 30.222 | 8.395 | 0.096 |
| 8 | 39 | 0 | 0 | 39 | 0 | 0 | 39 | 0 | 0 |
| 9 | 56 | 0 | 0 | 56 | 0 | 0 | 56 | 0 | 0 |
| $L$ | | 0.1176 | | | 0.0544 | | | 0.0402 | |

† $L'(s=5)=L'$ is just the lacunarity defined by Gefen *et al* [5], i.e. equation (1). $L'=0$ for figure 1(*c*) as we have mentioned before.

Gefen *et al* [5], denoted by $L'$

$$L'=(1/n)\sum_i (n_i-\bar{n})^2 \tag{1}$$

where $\bar{n}=\Sigma_i n_i/n$. $n$ is the number of square subarrays of $l\times l$ cells in an array of $b\times b$ cells. It can easily be seen that $n=(b-l+1)^2$, and $n_i$ represents the number of non-eliminated subsquares in the $i$th $l\times l$ 'covering'.

Unfortunately, equation (1) cannot cover the following important requirement: the lacunarity $L'=0$ if and only if the fractal is translationally invariant. A typical structure of a Sierpinski carpet is shown in figure 1(*c*), in which $b=9$, $l=5$. Only $5\times 5$ coverings containing nine eliminated subsquares are allowed; then $\bar{n}=n_i=16$ constantly and as a result $L'=0$. On the other hand, fractal 1(*c*) is obviously far from translationally invariant. There are many inhomogeneous Sierpinski carpets with $L'=0$. In figure 2

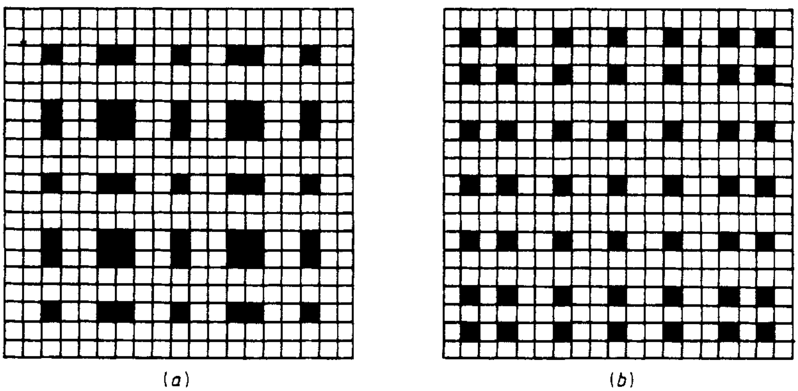

**Figure 2.** Two one-stage Sierpinski carpets with identical $b$ and $l$ values: $b=19$, $l=7$. (*a*) $L'=0$, improperly describes the inhomogeneity of the fractal, while $L=0.0202$, (*b*) $L'=3.265$, $L=0.0166$. $L$ values agree with the direct observation that figure 2(*b*) is more homogeneous than figure 2(*a*).

we show another such carpet with $b = 19$, $l = 7$. Table 2 sums the numerical results for figure 2. In addition one more simple example is added in figure 3.

The problem of improper $L' = 0$ comes from the special scale taken in equation (1): only $l \times l$ coverings are used. This $l$-scale definition possesses no physical necessity. We argue that the observation of lacunarity must be taking place in all scales larger than $l$ and all the $L$ values in the scales must become zero when the fractal becomes translationally invariant. With this consideration, we take the average of $L(s)$ over scales $l < s < b$, which is defined as follows, as the measurement of lacunarity:

$$L(s) = (1/\bar{n})\sqrt{L'(s)} \tag{2}$$

**Table 2.** Values of $\bar{n}$, $L'(s)$, $L(s)$ and $L$ for two $b = 19$, $l = 7$ Sierpinski carpets (figure 2). We see again $L'(s = 7) = L' = 0$ for figure 2($a$).

| | Figure 2($a$) | | | Figure 2($b$) | | |
|---|---|---|---|---|---|---|
| $S$ | $\bar{n}$ | $L'(s)$ | $L(s)$ | $\bar{n}$ | $L'(s)$ | $L(s)$ |
| 2 | 3.395 | 0.634 | 0.2345 | 3.395 | 0.239 | 0.1440 |
| 3 | 7.474 | 0.582 | 0.1020 | 7.751 | 0.270 | 0.0671 |
| 4 | 13.359 | 1.293 | 0.0851 | 13.750 | 1.188 | 0.0793 |
| 5 | 20.729 | 4.162 | 0.0984 | 21.760 | 1.062 | 0.0474 |
| 6 | 29.388 | 3.299 | 0.0618 | 31.408 | 1.140 | 0.0340 |
| 7 | 40 | 0 | 0 | 42.556 | 3.265 | 0.0425 |
| 8 | 52.889 | 4.988 | 0.0422 | 55.972 | 2.249 | 0.0268 |
| 9 | 67.107 | 10.708 | 0.0488 | 70.876 | 3.034 | 0.0246 |
| 10 | 82.360 | 5.670 | 0.0289 | 87.040 | 6.278 | 0.0288 |
| 11 | 100.247 | 10.309 | 0.0320 | 105.877 | 2.997 | 0.0164 |
| 12 | 199.000 | 25.250 | 0.0422 | 125.937 | 6.809 | 0.0207 |
| 13 | 137.959 | 15.264 | 0.0283 | 146.776 | 9.113 | 0.0206 |
| 14 | 160 | 0 | 0 | 171 | 0 | 0 |
| 15 | 186.560 | 12.326 | 0.0188 | 195.840 | 14.054 | 0.0191 |
| 16 | 213.750 | 21.188 | 0.0215 | 220 | 0 | 0 |
| 17 | 240 | 0 | 0 | 248.889 | 17.877 | 0.0170 |
| 18 | 275 | 0 | 0 | 275 | 0 | 0 |
| 19 | 312 | 0 | 0 | 312 | 0 | 0 |
| $L$ | | 0.0202 | | | 0.0166 | |

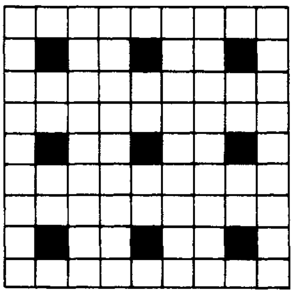

**Figure 3.** A simple example of inhomogeneous Sierpinski carpet with $b = 9$, $l = 3$. $L' = 0$ by equation (1), showing the failure of the equation.

where

$$L'(s) = (1/n) \sum_i (n_i - \bar{n})^2 = \overline{(\Delta n)^2}. \tag{3}$$

Here $S \times S$ coverings are used ($S$ varies from 1 to $b$) to calculate $L$. When $s = l$, equation (3) becomes identical with equation (1). We notice that $L'(s)$ represents the absolute deviation of $n_i$ from $\bar{n}$ and is of irregular variation with $S$ (see tables 1 and 2). $L(s)$ corresponds to the relative fluctuation of $n_i$ on $\bar{n}$. In the present discrete case, the smaller $S$ seems to make a larger fluctuation, then a larger $L(s)$. Our numerical results are also likely to exhibit such a tendency: $L(s)$ decreases with increasing $S$ (tables 1 and 2). However, more results are needed to confirm it.

Now a new approximate expression for lacunarity for a given $(b, l)$ Sierpinski carpet is given by

$$L = (1/N) \sum_{s=l}^{b} L(s) \tag{4}$$

where $N = b - l + 1$, and $L(s)$ is determined by equation (2). We abandon all $L(s)$ with $s < l$ because the discreteness of elimination will cause $L(s)$ to misbehave. Large scales $S$ have less effect on discreteness and can produce the lacunarity which better reflects the inhomogeneity of fractals. The values of $L$ from equation (4) can exhibit the uniformity of the eliminated squares distribution in Sierpinski carpets and ensure that $L = 0$ if and only if a fractal becomes a lattice with translationally invariant symmetry. Expression (4) also agrees with the conclusion: the lacunarity $L \to 0$ when $b \to \infty$ and the eliminated squares are scattered as uniformly as possible [4, 5].

Using expression (4), we calculate the values of $L$ for Sierpinski carpets shown in figures 1($a$)-($c$) and the numerical results are listed in table 1, which seem to be much more reasonable. The difficulty mentioned above has disappeared. We expect there are more results which will support our suggestion.

## References


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